

PRIME SPLITTINGS OF DETERMINANTAL IDEALS

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ABSTRACT. We consider determinantal facet ideals i.e., ideals generated by those minors of a generic matrix corresponding to facets of simplicial complexes. Using these ideals we give a combinatorial description for minimal primes of some mixed determinantal ideals. At the end, we discuss the minimal free resolution of these ideals.

INTRODUCTION

Let $X = (x_{ij})$ be a generic $m \times n$ matrix and $R = K[X]$ be the polynomial ring over the field K in indeterminates x_{ij} , and $m \leq n$. Each ideal J generated by a set of minors of X can be decomposed in several ways as $J = J_{\Delta_1, S_1} + J_{\Delta_2, S_2} + \cdots + J_{\Delta_r, S_r}$, where S_i is a subset of $[m]$, Δ_i is a simplicial complex on the vertex set $V_i \subset [n]$ of dimension less than $|S_i|$, and

$$J_{\Delta_i, S_i} = ([a_1 \dots a_k | b_1 \dots b_k] : \{b_1, \dots, b_k\} \in \mathcal{F}(\Delta_i) \text{ and } \{a_1, \dots, a_k\} \subset S_i)$$

for all i . Here $\mathcal{F}(\Delta_i)$ denotes the set of facets of Δ_i , and $[a_1 \dots a_k | b_1 \dots b_k]$ is the $k \times k$ minor of the submatrix of X with row indices a_1, \dots, a_k and column indices b_1, \dots, b_k . In this paper we study the components J_{Δ_i, S_i} . In order to simplify the notation we set J_Δ for $J_{\Delta, [m]}$, and $[b_1 \dots b_m]$ for $[1 \dots m | b_1 \dots b_m]$. The ideal J_Δ is called the determinantal facet ideal of Δ . It turns out that the simplicial complex Δ carries lots of algebraic information of the ideal J_Δ . Roughly speaking our goal is to translate some combinatorial properties of a simplicial complex into algebraic properties of its determinantal facet ideal.

The classical ideal I_k generated by all $k \times k$ minors of X , equals J_Δ , where Δ is the full $(k-1)$ -skeleton of the simplex on $[n]$. The ideal $I_k(X)$ has been extensively studied from the viewpoint of algebraic geometry and commutative algebra. The variety $\mathcal{V}(I_k)$ consists of all K -linear maps from K^m to K^n of rank less than k . It is well known that the coordinate ring $S/I_k(X)$ is a normal Cohen–Macaulay domain, see [2, 21]. Moreover, Sturmfels [23] and Caniglia et al. [7] showed that the set of all $k \times k$ minors of X forms a universal Gröbner basis of $I_k(X)$. Their technique provided a new proof of the Cohen–Macaulayness of the coordinate ring $S/I_k(X)$ and it has been used to compute numerical invariants of these rings, like the multiplicity and the Hilbert function, see [4, 9, 20]. Some excellent references on the theory of determinantal ideals are the book [6] of Bruns and Vetter, and the paper [2] of Bruns and Conca.

Several other classes of determinantal ideals that have been studied in the literature turn out to be determinantal facet ideals. For example, Herzog et al. in [18] studied determinantal ideals of a $2 \times n$ matrix which equal J_Δ , when Δ is a simple graph. They showed that many of the algebraic properties of these ideals can be translated into combinatorial properties of their corresponding graphs. These ideals arise naturally in the study

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of conditional independence ideals, see [17, 18]. Hosten and Sullivant in [22] considered the ideal generated by maximal adjacent minors of a generic matrix. This ideal is the determinantal ideal of the adjacent complex, i.e. the simplicial complex on $[n]$ with all subsets of the form $\{a, a+1, \dots, a+m-1\}$ for $1 \leq a \leq n-m+1$, as its facets. All minimal primes of these ideals can also be recognized as determinantal facet ideals for suitable simplicial complexes. For general *pure* simplicial complexes, determinantal facet ideals were introduced and studied in [12].

Motivated by geometrical considerations the more general class of ladder determinantal ideals have been considered by Conca, Gonciulea, and Miller, see [8, 14]. These ideals can be decomposed in a nice way as $J = J_{\Delta_1, S_1} + J_{\Delta_2, S_2} + \dots + J_{\Delta_r, S_r}$ such that J_{Δ_i, S_i} are all classical determinantal ideals with $S_r \subset \dots \subset S_2 \subset S_1$. This decomposition helps to show how these ideals share many properties with the classical determinantal ideals, see [8, 9].

For special cases we show how algebraic properties of J_{Δ_i, S_i} influence algebraic properties of $J = J_{\Delta_1, S_1} + J_{\Delta_2, S_2} + \dots + J_{\Delta_r, S_r}$. Indeed our motivation to study determinantal facet ideals is the following algebraic problem: given an ideal I as the sum of ideals I_1, \dots, I_r , how can one extract the algebraic invariants of I (e.g. Hilbert function, primary components, minimal free resolution, etc.) from the algebraic invariants of its subideals I_i ? In this generality the problem is hopeless, since each ideal can be written as a sum of principal ideals. Thus one can not expect a general result in this direction. However our examples (e.g. Examples 2.7 and 3.3) show that there exist ideals whose algebraic invariants can be better understood from studying their subideals. For example, motivated by our examples we realized that there exist ideals I with nice decompositions into smaller ideals I_1, \dots, I_r such that minimal primes of I can be determined from minimal primes of the ideals I_i . Our aim is to study and characterize some classes of ideals with this property.

Definition 0.1. Let I, I_1, \dots, I_r be some ideals such that $I = I_1 + \dots + I_r$. Then $I_1 + \dots + I_r$ is a *prime splitting* of I if the following condition holds:

- (*) P is a minimal prime ideal of I if and only if there exist minimal prime ideals P_{ℓ_i} of I_i such that $P = P_{\ell_1} + \dots + P_{\ell_r}$.

The paper is structured as follows. In Section 1 we present a property on the facets of Δ which guarantees that the generators of J_Δ form a Gröbner basis with respect to the lexicographical order. Simplicial complexes with this property are called *closed*, and we are able to compute the numerical invariants of ideals associated to closed simplicial complexes, see Corollary 1.7.

In Section 2 we introduce *block adjacent* simplicial complexes. We consider the ideal $J_\Delta = J_{\Delta_1} + \dots + J_{\Delta_r}$, where each Δ_i is an $(m-1)$ -dimensional simplicial complex on the vertex set $\{u_i, u_i+1, \dots, v_i\}$ with $u_1 < u_2 < \dots < u_r$ and for each i , the consequent complexes Δ_{i-1} and Δ_i intersect in the vertices $u_i, u_i+1, \dots, u_i+t_i-1$ for some $0 < t_i < m$. These determinantal facet ideals can be considered as a generalization of the ideal of maximal adjacent minors studied by Hosten and Sullivant in [22]. We give a combinatorial description of minimal primes for determinantal facet ideal a block adjacent simplicial complex. First, we reduce the problem to the case where the ideal contains *all* maximal adjacent minors of X . Then we show how to distinguish minimal primes of this ideal among the prime ideals constructed by Hosten and Sullivant in [22], see Theorem 2.3. In Theorem 2.8 we construct a class of mixed determinantal ideals which are prime, and then using this result we can interpret the primary decomposition of ideals associated to a *union of block adjacent* simplicial complexes. In fact in Theorem 2.6 we will give an explicit

prime splitting of J_Δ in terms of its ingredients block adjacent complexes. Describing the prime ideals of these ideals is a step to find the primary decomposition of determinantal facet ideals in general.

In Section 3 we study ideals associated to simplicial complexes which generalize forests. Let Δ_i be a union of block adjacent complexes for $i = 1, \dots, r$, and $\Delta = \Delta_1 \cup \dots \cup \Delta_r$ with some intersection properties. Then we consider the simple graph G_Δ whose vertices $1, \dots, r$ are identified with $\Delta_1, \dots, \Delta_r$, and two vertices are adjacent if and only if their corresponding subcomplexes have nonempty intersection. The question arises for which graphs, $J_{\Delta_1} + \dots + J_{\Delta_r}$ is a prime splitting of J_Δ . We study this question for forests, cycles, and cactus graphs, i.e., connected graphs in which each edge belongs to at most one cycle. In Theorem 3.2 we give a description of minimal primes for J_Δ in terms of minimal primes of the ideals J_{Δ_i} , where G_Δ is a forest. In particular, we show this description is indeed a prime splitting of J_Δ .

In the second part, we study the case when all Δ_i are full skeleton of some simplices. Then we associate a graph to Δ under a set of lighter conditions on its cliques, and we study the primality of J_Δ . This part can be considered as a generalization of the results of [12] in which Δ is assumed to be a pure simplicial complex. Note that in the first part we may have vertices in intersection of more than three complexes, but these vertices can be just in a subset of vertices. However in the second part, each vertex is in the intersection of at most two complexes, but each vertex can be an intersection vertex.

Section 4 is devoted to study minimal free resolutions of facet ideals associated to closed simplicial complexes. The minimal free resolution of the classical determinantal ideal generated by all maximal minors of X , is given by the Eagon-Northcott complex (see e.g. [10, A2.6.1]). In Theorem 4.7 we give a construction of the minimal free resolution of a determinantal facet ideal by applying a criterion given by Bruns and Conca in [2]. This leads us to the fact that the multigraded Betti numbers of these ideals are equal to the multigraded Betti numbers of their initial ideals with respect to the lexicographical term order, see Corollary 4.9. Finally our results together with a result of Bernstein and Zelevinsky from [1] imply that the generators of a determinantal facet ideal of a pure simplicial complex form a universal Gröbner basis if and only if the underlying simplicial complex is a full skeleton of a simplex.

1. DETERMINANTAL FACET IDEALS

1.1. Clique decomposition of Δ . First we attempt to decompose Δ into a union of its subcomplexes which are full skeletons of some simplices. This is a natural choice in order to decompose J_Δ into smaller ideals, since the determinantal ideal of a full skeleton of a simplex is a classical determinantal ideal which is well-understood from the algebraic point of view.

Let Γ be the full k -skeleton of a simplex on a subset of $[n]$. Then Γ is called a *clique* of Δ if $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$. A *maximal clique* of Δ is a clique which is not a subcomplex of another clique of Δ (i.e. it is maximal with respect to inclusion). Let $\Delta_1, \dots, \Delta_r$ be the maximal cliques of Δ . Then Δ is the union of its cliques and $\Delta = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_r$ is called the *clique decomposition* of Δ . For example, let Δ be the 2-dimensional simplicial complex on the vertex set $[9]$ with facets $F_1 = \{1, 2, 3\}$, $F_2 = \{1, 2, 4\}$, $F_3 = \{1, 3, 4\}$, $F_4 = \{2, 3, 4\}$, $F_5 = \{3, 4, 5\}$, $F_6 = \{5, 6, 7\}$, $F_7 = \{7, 8\}$, $F_8 = \{8, 9\}$ and $F_9 = \{7, 9\}$. Then Δ has the clique decomposition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, where Δ_1 is the 2-clique on the

vertex set $\{1, 2, 3, 4\}$, Δ_2 is the 2-clique on the vertex set $\{3, 4, 5\}$, Δ_3 is the 2-clique on the vertex set $\{5, 6, 7\}$, and Δ_4 is the 1-clique on the vertex set $\{7, 8, 9\}$.

1.2. Ideals whose generators form a Gröbner basis. Let $<_{\text{lex}}$ be the lexicographical order induced by the natural order of indeterminates

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{2n} > \cdots > x_{mn},$$

of matrix row by row from left to right. Here we study the question when the generators of J_Δ form a Gröbner basis of J_Δ with respect to $<_{\text{lex}}$. In order to explain this result, we need the following observation.

Remark 1.1. Assume that $\{b_1, b_2, \dots, b_t\}$ is a facet of Δ , and $1 \leq a_1 < \cdots < a_t \leq m$ are some integers. Then the initial term of $[a_1 \dots a_t | b_1 \dots b_t]$ is $x_{a_1 b_1} \cdots x_{a_t b_t}$. Therefore for each index b_k the variable x_{ib_k} appears in the support of an initial term of a minor $[a_1 \dots a_t | b_1 \dots b_t]$ if and only if $k \leq i \leq m - t + k$.

Definition 1.2. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ be the clique decomposition of Δ . Then Δ is called *closed* if for every pair of facets $\{b_1, b_2, \dots, b_t\} \in \mathcal{F}(\Delta_i)$ and $\{c_1, c_2, \dots, c_s\} \in \mathcal{F}(\Delta_j)$ with $i \neq j$ and $t \leq s$, and for all k we have

$$b_k \neq c_\ell, \quad \text{where} \quad \max\{1, k - m + s\} \leq \ell \leq m - t + k.$$

Example 1.3. Let Δ be the simplicial complex as Figure 1 (a). Then Δ is a closed simplicial complex with respect to the labeling given in Figure 1 (b), but it is not closed with respect to the labeling given in Figure 1 (c). The facets of Δ with respect to the labeling (b) are $\{1, 2\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$, and $\{6, 7\}$. Then Δ has the clique decomposition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, where Δ_1 is the 1-clique on the vertex set $\{1, 2\}$, Δ_2 is the 2-clique on the vertex set $\{2, 3, 4\}$, Δ_3 is the 2-clique on the vertex set $\{4, 5, 6\}$, and Δ_4 is the 1-clique on the vertex set $\{6, 7\}$. Δ is closed with respect to this labeling, since for example comparing the first two cliques of Δ , one observes that the vertex 1 of the facet $\{1, 2\}$ does not take the first and the second position in the facet $\{2, 3, 4\}$, and the vertex 2 does not have the second or third position in the facet $\{2, 3, 4\}$.

But Δ is not closed with respect to the labeling given in Figure 1 (c). Since the vertex 1 takes the first position in two facets $\{1, 2\}$ and $\{1, 3, 4\}$, and the vertex 5 takes the first position in the facet $\{5, 7\}$, and the second position in the facet $\{4, 5, 6\}$. However the simplicial complex is closed, since one may find a labeling on its vertices with respect to which Δ is closed.

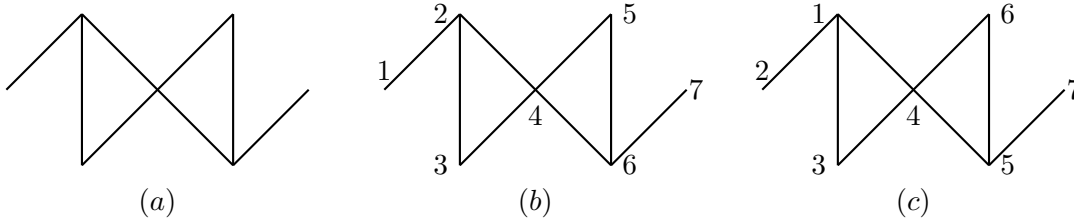


FIGURE 1.

Remark 1.4. Note that if Δ is a closed simplicial complex, i.e. there exists a labeling on the vertices of Δ such that Δ is closed, then corresponding to each labeling on the vertices, we can find a lexicographical term order $<$ such that the generators form a Gröbner basis

with respect to $<$. For example corresponding to the labeling given in Figure 1 (c) one can consider the lexicographical term order induced by

$$x_{12} > x_{11} > x_{13} > x_{14} > x_{16} > x_{15} > x_{17} > x_{22} > \cdots > x_{37}.$$

Now one can easily see that the generators form a Gröbner basis for J_Δ with respect to $<$. Therefore with no loss of generality we can always consider a labeling on the vertices of a closed simplicial complex Δ such that the generators form a Gröbner basis with respect to the lexicographical term order induced by $x_{11} > \cdots > x_{1n} > \cdots > x_{mn}$.

Theorem 1.5. *If Δ is a closed simplicial complex, then the set of the generators of J_Δ forms a Gröbner basis for J_Δ with respect to $<_{\text{lex}}$, and J_Δ is a radical ideal.*

Proof. We show that all S -pairs, $S([a_1 \dots a_t | b_1 \dots b_t], [c_1 \dots c_s | d_1 \dots d_s])$ reduce to zero. Assume that $\{b_1, \dots, b_t\} \in \Delta_i$ and $\{d_1, \dots, d_s\} \in \Delta_j$. If $i \neq j$, then $\text{in}_<[a_1 \dots a_t | b_1 \dots b_t]$ and $\text{in}_<[c_1 \dots c_s | d_1 \dots d_s]$ have no common factor which implies that their S -polynomial reduces to zero. Now assume that $i = j$. Then $s = t$, and since Δ is closed, all s -subsets of $\{b_1, \dots, b_s\} \cup \{d_1, \dots, d_s\}$ belong to Δ . Therefore $S([a_1 \dots a_s | b_1 \dots b_s], [c_1 \dots c_s | d_1 \dots d_s])$ reduces to zero with respect to $s \times s$ minors of X , where column indices are among $b_1, \dots, b_s, d_1, \dots, d_s$. Then the assertion follows by applying Buchberger criterion. Since generators of the initial ideal of J_Δ are all squarefree, we deduce that J_Δ is radical. \square

Remark 1.6. In terms of initial monomials, the *closed* property guarantees that the initial terms $\text{in}_<[a_1 \dots a_t | b_1 \dots b_t]$ and $\text{in}_<[a'_1 \dots a'_s | c_1 \dots c_s]$ are *relatively prime* for all facets $\{b_1, \dots, b_t\}$ and $\{c_1, \dots, c_s\}$ of distinct cliques of Δ , and all integers $1 \leq a_1 < \dots < a_t \leq m$, and $1 \leq a'_1 < \dots < a'_s \leq m$. As we see in Example 1.3 with respect to the first labeling on $V(\Delta)$, the initial terms of each pair of the elements in the generating set of J_Δ are relatively prime. For example $\text{in}_<[a_1 a_2 | 12] = x_{a_1 1} x_{a_2 2}$ and $\text{in}_<[123 | 234] = x_{12} x_{23} x_{34}$ are relatively prime for all $1 \leq a_1 < a_2 \leq 3$, while in the second case $\text{in}_<[12 | 12] = x_{11} x_{22}$ and $\text{in}_<[123 | 134] = x_{11} x_{23} x_{34}$ which are not relatively prime, and $\text{in}_<[23 | 57] = x_{25} x_{37}$ and $\text{in}_<[123 | 456] = x_{14} x_{25} x_{36}$ which are not relatively prime either.

A consequence of Theorem 1.5 is the following result which follows by the same argument given in the proof of [12, Corollary 1.3].

Corollary 1.7. *Let $\Delta = \Delta_1 \cup \dots \cup \Delta_r$ be the clique decomposition of the closed simplicial complex Δ of dimension $m - 1$, $t_\ell = \dim(\Delta_\ell)$ and $n_\ell = |V(\Delta_\ell)|$ for all ℓ . Then:*

- (1) $\text{height } J_\Delta = \sum_{\ell=1}^r \text{height } J_{\Delta_\ell} = \sum_{\ell=1}^r (n_\ell - t_\ell),$
- (2) J_Δ is Cohen-Macaulay,
- (3) The Hilbert series of R/J_Δ has the form

$$H_{R/J_\Delta}(t) = \prod_{\ell=1}^r H_{R/J_{\Delta_\ell}}(t),$$

- (4) The multiplicity of R/J_Δ is

$$e(R/J_\Delta) = \prod_{\ell=1}^r e(R/J_{\Delta_\ell}) = \prod_{\ell=1}^r \binom{n_\ell}{t_\ell}.$$

Proof. By Remark 1.6 we have that the initial ideals $\text{in}_<(J_{\Delta_\ell})$ are monomial ideals in disjoint sets of variables and

$$(1.1) \quad R/\text{in}_<(J_\Delta) \cong \bigotimes_{i=1}^r R_i/\text{in}_<(J_{\Delta_i}),$$

where R_i are polynomial rings in disjoint sets of variables whose union is the set of all the variables of X . This fact together with the known formula for the height of the determinantal ideals (see e.g. [11, Theorem 6.35]) implies (1). It is known that all $R_i/\text{in}_{<}(J_{\Delta_i})$ are Cohen-Macaulay (see e.g. [7] and [23]). Therefore by (1.1) we get that $R/\text{in}_{<}(J_{\Delta})$ is Cohen-Macaulay, and so R/J_{Δ} is Cohen-Macaulay as well, (see e.g. [16, Corollary 3.3.5]).

By [16, Corollary 3.3.5] modules R/J_{Δ} and $R/\text{in}_{<}(J_{\Delta})$ have the same Hilbert series. The Hilbert series and the multiplicity of the classical determinantal ring generated by maximal minors of X are known (see e.g. [9, Corollary 1] or [2, Theorem 6.9] and [20, Theorem 3.5]). Therefore equation (1.1) implies statements (3) and (4). \square

1.3. Gröbner basis of a class of mixed determinantal ideals. Here we study a class of mixed determinantal ideals which enjoy the property that their generators form a Gröbner basis with respect to $<_{\text{lex}}$. Then this property enables us to use the advantage of the Gröbner basis theory in order to find nonzerodivisors modulo determinantal facet ideals. It turns out that minimal primes of a determinantal facet ideal are all of this type.

We first fix our notation. Let $S = \{i_1 < \dots < i_k\} \subseteq [m]$ and $B = \{j_1 < \dots < j_t\} \subseteq [n]$. We denote by $X_S[B]$ the submatrix of X with row indices i_1, \dots, i_k , and column indices j_1, \dots, j_t in X . If $S = [m]$, then we simplify the notation to $X[B]$. In order to prove our main result of this section, we need the following technical lemma.

Lemma 1.8. *Let F be the set of maximal minors of the submatrix $X_S[b_1, \dots, b_t]$ of X , and G be the set of maximal minors of the submatrix $X_{S'}[d_1, \dots, d_k]$, where $k \leq |S'|$ and $k < t$. Then the set $F \cup G$ forms a Gröbner basis for the ideal which they generate with respect to $<_{\text{lex}}$ if one of the following holds:*

- (i) $S \subseteq S'$ and $t \leq |S|$,
- (ii) $S' \subseteq S$ and $\{d_1, \dots, d_k\} \subset \{b_1, \dots, b_t\}$.

Proof. The case $S' \subseteq S$, $\{d_1, \dots, d_k\} \subset \{b_1, \dots, b_t\}$ and $|S| < t$, is a consequence of [22, Lemma 4.2]. So we assume that $t \leq |S|$. It is known that the set F forms a Gröbner bases for the ideal $I = \langle F \rangle$, and G forms a Gröbner bases for the ideal $J = \langle G \rangle$. By [22, Lemma 4.1] in order to prove the statement, we show that for arbitrary minors $f = [a_1 \dots a_t | b_1 \dots b_t] \in F$ and $g = [c_1 \dots c_k | d_1 \dots d_k] \in G$ there exists an element $h \in I \cap J$ with $\text{in}(h) = \text{LCM}(\text{in}(f), \text{in}(g))$. We have $\text{in}(f) = x_{a_1 b_1} x_{a_2 b_2} \dots x_{a_t b_t}$ and $\text{in}(g) = x_{c_1 d_1} x_{c_2 d_2} \dots x_{c_k d_k}$. Assume that $X = X_S[b_1, \dots, b_t]$ and $Y = X_{S'}[d_1, \dots, d_k]$. Let Y_1 be the set of columns of Y indexed by d_j with $x_{a_\ell b_\ell} = x_{c_j d_j}$ for some ℓ , and let Y'_2 be the subset of $\{b_1, \dots, b_t\}$ consisting those indices not appeared in Y_1 . Let B be the submatrix of X with row indices a_1, \dots, a_t and the columns index set Y'_2 .

Assume that $|\{d_1, \dots, d_k\} - |Y_1|| = s$, and the columns d_{j_u} are not in Y_1 for $u = 1, \dots, s$. Let A be the $s \times k$ submatrix of X with the column indices d_1, \dots, d_k , and the row indices c_{j_1}, \dots, c_{j_s} . Let Y' be the submatrix $X_S[d_1, \dots, d_k]$ of X . Then we define an $(s+t) \times (k+t-|Y_1|)$ matrix

$$H = \begin{pmatrix} A & 0 \\ Y' & B \end{pmatrix}.$$

Now in case (i), the determinant of H can be computed by the Laplace expansion using the $k \times k$ minors of the first k columns, which implies that $|H|$ can be generated by the determinants of $k \times k$ submatrices of the matrix $\begin{pmatrix} A \\ Y' \end{pmatrix}$ which are all in J_{Δ} , since Δ is closed. Computing the Laplace expansion using the first k columns of the matrix H also shows that $\text{in}(h) = \text{LCM}(\text{in}(f), \text{in}(g))$.

In case (ii) the determinant of H can be computed by the Laplace expansion using the maximal minors of the last t rows of the matrix which are all among the minors of $X_S[b_1, \dots, b_t]$. This also implies that $\text{in}(h) = \text{LCM}(\text{in}(f), \text{in}(g))$, as desired. \square

Here we describe the idea of the proof of Lemma 1.8 for case (i), in an example, where $n = 6$, $t = 5$ and $k = 4$.

Example 1.9. Suppose that $f = [12356|13456]$ and $g = [1345|2456]$. As we see $\text{in}(f) = x_{11}x_{23}x_{34}x_{55}x_{66}$ and $\text{in}(g) = x_{12}x_{34}x_{45}x_{56}$. Then $Y_1 = \{4\}$, $Y_2' = \{1, 3, 5, 6\}$ and so we have $A = X_{\{1,4,5\}}[2, 4, 5, 6]$, and $B = X_{\{1,2,3,5,6\}}[1, 3, 5, 6]$ as

$$A = \begin{pmatrix} x_{12} & x_{14} & x_{15} & x_{16} \\ x_{42} & x_{44} & x_{45} & x_{46} \\ x_{52} & x_{54} & x_{55} & x_{56} \end{pmatrix}, \quad B = \begin{pmatrix} x_{11} & x_{13} & x_{15} & x_{16} \\ x_{21} & x_{23} & x_{25} & x_{26} \\ x_{31} & x_{33} & x_{35} & x_{36} \\ x_{51} & x_{53} & x_{55} & x_{56} \\ x_{61} & x_{63} & x_{65} & x_{66} \end{pmatrix}$$

which shows that

$$H = \begin{pmatrix} \underline{x_{12}} & x_{14} & x_{15} & x_{16} & 0 & 0 & 0 & 0 \\ x_{42} & x_{44} & \underline{x_{45}} & x_{46} & 0 & 0 & 0 & 0 \\ x_{52} & x_{54} & x_{55} & \underline{x_{56}} & 0 & 0 & 0 & 0 \\ x_{12} & x_{14} & x_{15} & x_{16} & \underline{x_{11}} & x_{13} & x_{15} & x_{16} \\ x_{22} & x_{24} & x_{25} & x_{26} & x_{21} & \underline{x_{23}} & x_{25} & x_{26} \\ x_{32} & \underline{x_{34}} & x_{35} & x_{36} & x_{31} & x_{33} & x_{35} & x_{36} \\ x_{52} & x_{54} & x_{55} & x_{56} & x_{51} & x_{53} & \underline{x_{55}} & x_{56} \\ x_{62} & x_{64} & x_{65} & x_{66} & x_{61} & x_{63} & x_{65} & \underline{x_{66}} \end{pmatrix}.$$

It is easy to see that $\text{in}(h) = \text{LCM}(\text{in}(f), \text{in}(g))$ using the Laplace expansion along the first four columns of the matrix H .

Theorem 1.10. *Let $\Delta = \Delta_1 \cup \Delta_2$ be the clique decomposition of the closed simplicial complex Δ . Then the minimal generating set of the ideal*

$$I = J_{\Delta_1} + J_{\Delta_2} + J_{\Delta_1 \cap \Delta_2}$$

forms a Gröbner basis for I . In particular, I is a radical ideal.

Proof. First note that the initial terms of the elements of J_{Δ_1} , and the initial terms of the elements of J_{Δ_2} are relatively prime. Using this fact together with Lemma 1.8 we conclude that the condition of [22, Lemma 4.1] holds. This implies the first statement. Since the initial terms of the generators of I with respect to the lexicographic order are all squarefree monomials, we have that I is radical. \square

2. DETERMINANTAL FACET IDEALS OF BLOCK ADJACENT SIMPLICIAL COMPLEXES

Let $\Delta = \Delta_1 \cup \dots \cup \Delta_r$, where each Δ_i is an $(m-1)$ -dimensional simplicial complex on the vertex set $\{u_i, u_i+1, \dots, v_i\}$ such that $u_1 < u_2 < \dots < u_r$ and $V(\Delta_{i-1}) \cap V(\Delta_i) = \{u_i, u_i+1, \dots, u_i+t_i-1\}$ for some $0 < t_i < m$. One can easily observe that these complexes are all closed. These determinantal facet ideals can be considered as a generalization of the ideal of maximal adjacent minors studied by Hosten and Sullivant in [22]. We first consider the case where $t_i = m-1$ for all i , i.e., each pair of consequent complexes Δ_{i-1} and Δ_i intersect in $m-1$ vertices. These simplicial complexes are called *block adjacent*. Then we turn to the general case in which $0 < t_i < m$ for all i , i.e., Δ is a *union* of block

adjacent complexes. Throughout we assume that $\text{char}(K) = 0$, and the set $F = \{i_1, \dots, i_t\}$ is denoted by $i_1 i_2 \dots i_t$, where $i_1 < i_2 < \dots < i_t$.

As an example, consider the simplicial complex Δ with $\mathcal{F}(\Delta) = \{123, 234, 235, 245, 345, 456\}$. One can write Δ as $\Delta_1 \cup \Delta_2 \cup \Delta_3$, where Δ_1 is the simplex on the vertex set $\{1, 2, 3\}$, Δ_2 is the 2-skeleton of the simplex on the vertex set $\{2, 3, 4, 5\}$, and Δ_3 is the simplex on the vertex set $\{4, 5, 6\}$. Therefore Δ is a block adjacent simplicial complex.

2.1. Block adjacent simplicial complex and prime sequences. Let $\Delta = \Delta_1 \cup \dots \cup \Delta_r$ be a block adjacent simplicial complex on the vertex set $[n]$ such that $V(\Delta_{i-1}) \cap V(\Delta_i) = \{u_i, u_i + 1, \dots, u_i + m - 2\}$ for all i . In the following we describe the minimal primes for J_Δ in terms of determinantal facet ideals of some other complexes.

Definition 2.1. The partition Γ of $[n]$ is called a *prime sequence* of Δ if

$$\Gamma : [a_1, b_1], [a_2, b_2], \dots, [a_t, b_t]$$

is a sequence of intervals with the following properties:

- (1) $1 = a_1 < a_2 < \dots < a_t$ and $b_1 < b_2 < \dots < b_t = n$;
- (2) $b_\ell - a_\ell \geq m - 1$ for $\ell = 1, t$ and $b_\ell - a_\ell \geq m$ for $2 \leq \ell \leq t - 1$;
- (3) $0 \leq b_\ell - a_{\ell+1} \leq m - 2$ for all ℓ ;
- (4) if $|V(\Delta_i)| > m$ for some i , then there exists ℓ with $V(\Delta_i) \subseteq [a_\ell, b_\ell]$.

Now, we consider the ideal P_Γ of R which is generated by

- (i) all maximal $m \times m$ minors of the submatrix $X[a_\ell, b_\ell]$ for $\ell = 1, \dots, t$, and
- (ii) all maximal $(b_\ell - a_{\ell+1} + 1) \times (b_\ell - a_{\ell+1} + 1)$ minors of the submatrix $X[a_{\ell+1}, b_\ell]$ for $\ell = 1, \dots, t - 1$.

Indeed $P_\Gamma = J_{\Delta_\Gamma}$, where Δ_Γ is the union of $(m - 1)$ -dimensional simplices on the vertex sets $[a_i, b_i]$, and $(b_\ell - a_{\ell+1})$ -skeletons of simplices on the vertex sets $[a_{\ell+1}, b_\ell]$.

Example 2.2. Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{123, 124, 134, 234, 345, 456, 567\}$. One can observe that Δ is a block adjacent complex with the following prime sequences:

- $\Gamma : [1, 7]$
- $\Gamma : [1, 6], [5, 7]$
- $\Gamma : [1, 5], [5, 7]$
- $\Gamma : [1, 5], [4, 7]$
- $\Gamma : [1, 4], [4, 7]$
- $\Gamma : [1, 4], [3, 7]$
- $\Gamma : [1, 4], [3, 6], [5, 7]$.

In the following the vector space generated by vectors corresponding to columns $a, a + 1, \dots, b$ of X is denoted by $\text{span}[a, b]$, and the maximum number of linearly independent column vectors $a, a + 1, \dots, b$ of X , i.e., rank of the submatrix $X[a, b]$ is denoted by $\text{rk}[a, b]$. We denote the set of all prime sequences of Δ by \mathcal{A}_Δ .

Theorem 2.3. The minimal prime decomposition of J_Δ is given by

$$J_\Delta = \bigcap_{\Gamma \in \mathcal{A}_\Delta} P_\Gamma.$$

Proof. The block adjacent simplicial complex $\Delta = \Delta_1 \cup \dots \cup \Delta_r$ is closed, and so by Theorem 1.5 the ideal J_Δ is radical. Therefore it is enough to show that $\mathcal{V}(J_\Delta) = \bigcup_{\Gamma \in \mathcal{A}_\Delta} \mathcal{V}(P_\Gamma)$. The case that $|V(\Delta_i)| = m$ for all i , holds by [22, Theorem 3.5]. Assume that there exists at least one subcomplex with $|V(\Delta_\ell)| > m$. Let X be a matrix in $\mathcal{V}(J_\Delta)$. Then by [22,

Lemma 3.4] there exists a sequence $\Gamma = \{[a_1, b_1], \dots, [a_k, b_k]\}$ such that $X \in \mathcal{V}(P_\Gamma)$ and Γ has properties (1), (2) and (3). Assume that $\Delta_{i_1}, \dots, \Delta_{i_p}$ are the subcomplexes of Δ such that $|V(\Delta_{i_j})| > m$ and $V(\Delta_{i_j})$ is not the subset of any interval of Γ . Then we will construct a sequence Γ' with properties (1), (2), and (3) such that $X \in \mathcal{V}(P_{\Gamma'})$, $V(\Delta_{i_1})$ is the subset of an interval of Γ' and *just* for complexes $\Delta_{i_2}, \dots, \Delta_{i_p}$, the vertex set is not contained in any interval of Γ' . Then continuing this argument we will get a prime sequence for Δ such that the variety of its associated ideal contains X . Note that the minimality of the constructed prime ideals follows by Corollary 4.5 and Corollary 4.6 of [22] which completes the proof.

Assume that $V(\Delta_{i_1}) = [u, v]$. Therefore there exist integers s and t such that

$$[u, v] \subseteq [a_t, b_t] \cup [a_{t+1}, b_{t+1}] \cup \dots \cup [a_s, b_s],$$

where $a_t < u < b_t$ or $a_{t+1} = b_t = u$, and $a_s < v < b_s$ or $b_{s-1} = a_s = v$. We will consider the following four cases:

Case 1. If $v - a_s + 1 < m$ and $b_t - u + 1 < m$, then we consider the sequence

$$\Gamma_1 : [a_1, b_1], \dots, [a_t, b_t], [u, v], [a_s, b_s], \dots, [a_k, b_k].$$

Note that all $m \times m$ minors of $X[u, v]$ are zero, since $X \in \mathcal{V}(J_\Delta) \subset \mathcal{V}(J_{\Delta_\ell})$. Also all maximal minors of the submatrices $X[u, b_t]$ and $X[a_s, v]$ are zero, since $[a_{t+1}, b_t] \subseteq [u, b_t]$, $[a_s, b_{s-1}] \subseteq [a_s, v]$ and by our assumption that $X \in P_\Gamma$ we have all maximal minors of $X[a_{t+1}, b_t]$ and $X[a_s, b_{s-1}]$ are zero. Therefore $X \in \mathcal{V}(P_{\Gamma_1})$, as desired.

Case 2. Let $v - a_s + 1 \geq m$ and $b_t - u + 1 < m$. If $v = n$, then the sequence

$$\Gamma' : [a_1, b_1], \dots, [a_t, b_t], [u, v]$$

has desired properties by the same argument as Case 1. Assume that $v < n$. If $\text{rk}[a_s, v] = m - 1$, then $\text{span}[a_s, b_s] = \text{span}[a_s, v] = \text{span}[u, v] = \text{span}[u, b_s]$. This shows that all $m \times m$ minors of $X[u, b_s]$ are zero. Moreover, $[a_{t+1}, b_t] \subseteq [u, b_t]$ implies that all maximal minors of $X[u, b_t]$ are zero, since $X \in \mathcal{V}(P_\Gamma)$. Thus the sequence

$$\Gamma_2 : [a_1, b_1], \dots, [a_t, b_t], [u, b_s], [a_{s+1}, b_{s+1}], \dots, [a_k, b_k]$$

has desired properties.

Assume that $\text{rk}[a_s, v] < m - 1$. If $b_s > v + 1$, then we define Γ'_2 as

$$\Gamma'_2 : [a_1, b_1], \dots, [a_t, b_t], [u, v], [v - (m - 2), b_s], [a_{s+1}, b_{s+1}], \dots, [a_k, b_k].$$

Note that the width of the interval $[v - (m - 2), b_s]$ is greater than m and the width of the interval $[v - (m - 2), v]$, i.e. the intersection of intervals $[u, v]$ and $[v - (m - 2), b_s]$ is $m - 1$. These facts, together with the above condition on $\text{rk}[a_s, v]$ show that the constructed sequence Γ'_2 has desired properties.

Now assume that $b_s = v + 1$. If $b_s = n$, then Γ'_2 has desired properties. Otherwise $[a_{s+1}, b_{s+1}]$ is among the intervals Γ . Now we should consider two different subcases:

Subcase 2.1. Let $a_{s+1} < v$. If $\text{rk}[a_{s+1}, v] < v - a_{s+1} + 1$, then the sequence

$$\Gamma_{2.1} : [a_1, b_1], \dots, [a_t, b_t], [u, v], [a_{s+1}, b_{s+1}], [a_{s+2}, b_{s+2}], \dots, [a_k, b_k].$$

fulfills our conditions. If $\text{rk}[a_{s+1}, v] = v - a_{s+1} + 1$, then $v + 1^{\text{th}}$ column belongs to the $\text{span}[a_{s+1}, v]$ which is the subset of the $\text{span}[u, v]$. Therefore all $m \times m$ minors of $X[u, v + 1]$ are zero which implies that the following sequence fulfills our conditions:

$$\Gamma_{2.2} : [a_1, b_1], \dots, [a_t, b_t], [u, v + 1], [a_{s+1}, b_{s+1}], [a_{s+2}, b_{s+2}], \dots, [a_k, b_k].$$

Subcase 2.2. Let $a_{s+1} \geq v$. If $a_{s+1} = v + 1$, then the variables of the $v + 1^{\text{th}}$ column of X are in P_Γ , and so all $m \times m$ minors of $X[u, b_s]$ are zero. Hence the sequence $\Gamma_{2.2}$ has again desired properties. Now, assume that $a_{s+1} = v$. If the v^{th} column of X is nonzero, then $v + 1^{\text{th}}$ column is a multiplication of the v^{th} column and so it belongs to the $\text{span}[u, v]$, since $\text{rk}[v, v + 1] = 1$. Therefore $\Gamma_{2.2}$ fulfills our conditions. Otherwise, the following sequence has desired properties:

$$\Gamma_{2.3} : [a_1, b_1], \dots, [a_t, b_t], [u, v], [v, b_{s+1}], [a_{s+2}, b_{s+2}], \dots, [a_k, b_k].$$

Case 3. Let $v - a_s + 1 < m$ and $b_t - u + 1 \geq m$. If $u = 1$, then by the same argument as Case 1 the sequence

$$\Gamma' : [u, v], [a_s, b_s], \dots, [a_k, b_k]$$

has desired property. Suppose that $u > 1$. If $\text{rk}[u, b_t] = m - 1$, then $\text{span}[a_t, b_t] = \text{span}[u, b_t] = \text{span}[u, v] = \text{span}[a_t, v]$. Then $\text{rk}[a_t, v] < m$ shows that all $m \times m$ minors of $X[a_t, v]$ are zero, and so the following sequence has desired properties:

$$\Gamma_3 : [a_1, b_1], \dots, [a_{t-1}, b_{t-1}], [a_t, v], [a_s, b_s], \dots, [a_k, b_k].$$

Now assume that $\text{rk}[u, b_t] < m - 1$. If $a_t < u - 1$, then we consider the sequence

$$\Gamma'_3 : [a_1, b_1], \dots, [a_t, u + (m - 2)], [u, v], [a_s, b_s], [a_{s+1}, b_{s+1}], \dots, [a_k, b_k].$$

The width of the interval $[a_t, u + (m - 2)]$ is greater than m , and the width of $[u, u + (m - 2)]$, i.e. the intersection of the intervals $[u, v]$ and $[a_t, u + (m - 2)]$ is $m - 1$. Hence our condition on $\text{rk}[u, b_t]$ guarantees that Γ'_3 fulfills desired properties.

Let $a_t = u - 1$. If $a_t = 1$, then Γ'_3 has desired properties. Assume that $a_t = u - 1 > 1$. So we have $[a_{t-1}, b_{t-1}] \in \Gamma$. Now two different subcases should be considered:

Subcase 3.1. Let $b_{t-1} > u$. If $\text{rk}[u, b_{t-1}] < b_{t-1} - u + 1$, then the sequence

$$\Gamma_{3.1} : [a_1, b_1], \dots, [a_{t-1}, b_{t-1}], [u, v], [a_s, b_s], \dots, [a_k, b_k]$$

has desired properties. If $\text{rk}[u, b_{t-1}] = b_{t-1} - u + 1$, then we consider the sequence

$$\Gamma_{3.2} : [a_1, b_1], \dots, [a_{t-1}, b_{t-1}], [u - 1, v], [a_s, b_s], \dots, [a_k, b_k].$$

Note that $u - 1^{\text{th}}$ column belongs to the $\text{span}[u, b_{t-1}]$. This together with the fact that $\text{span}[u, b_{t-1}] \subseteq \text{span}[u, v]$ implies that the $u - 1^{\text{th}}$ column belongs to the $\text{span}[u, v]$ and so all $m \times m$ minors of $X[u - 1, v]$ are zero.

Subcase 3.2. Let $b_{t-1} \leq u$. If $b_{t-1} = u - 1$, then the variables corresponding to the $u - 1^{\text{th}}$ column of X are all in P_Γ , and so all $m \times m$ minors of $X[a_{t-1}, v]$ are zero. Hence the sequence $\Gamma_{3.2}$ has desired properties.

Let $b_{t-1} = u$. If the u^{th} column of X is nonzero, then $u - 1^{\text{th}}$ column is a multiplication of the u^{th} column and so it belongs to the $\text{span}[u, v]$, since $\text{rk}[u - 1, u] = 1$. Therefore $\Gamma_{3.2}$ has desired properties. Otherwise, the sequence

$$\Gamma_{3.3} : [a_1, b_1], \dots, [a_{t-1}, b_{t-1}], [u, v], [a_s, b_s], \dots, [a_k, b_k]$$

has desired properties.

Case 4. If $v - a_s + 1 \geq m$ and $b_t - u + 1 \geq m$, then by combination of the arguments given in Case 2 and Case 3, we can construct the proper prime sequences. \square

2.2. A union of block adjacent simplicial complexes. Here we study a class of simplicial complexes which are unions of block adjacent simplicial complexes, i.e., consequent complexes Δ_{i-1} and Δ_i in decomposition of Δ , intersect in t_i vertices for $0 < t_i < m$. We will describe minimal primes of determinantal facet ideals of these complexes. First we consider a more compact decomposition of Δ in which $\Delta = \Delta_1 \cup \dots \cup \Delta_r$, where each Δ_i is a block adjacent complex studied in the previous subsection.

Definition 2.4. Let $\Delta = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_r$, where Δ_i is a block adjacent simplicial complex on the vertex set $[u_i, v_i]$ for each i , and $v_{i-1} - m + 3 \leq u_i \leq v_{i-1}$. The partition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_r$ of $[u_1, v_r]$ is called a *prime sequence* if each partition

$$\Gamma_i = \{[a_{i1}, b_{i1}], [a_{i2}, b_{i2}], \dots, [a_{it_i}, b_{it_i}]\}$$

is a prime sequence of Δ_i for $i = 1, \dots, r$. \mathcal{A}_Δ denotes the set of all prime sequences of Δ .

Example 2.5. Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{123, 345, 456, 467, 457, 567, 678, 789\}$. One can observe that $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1 is the adjacent simplicial complex on the vertex set $\{1, 2, 3\}$ with $\mathcal{F}(\Delta_1) = \{123\}$ and Δ_2 is a block adjacent simplicial complex on the vertex set $\{3, 4, \dots, 9\}$ with $\mathcal{F}(\Delta_2) = \{345, 456, 467, 457, 567, 678, 789\}$. Then the prime sequences of Δ are

- $\Gamma = \{[1, 3]\} \cup \{[3, 9]\}$
- $\Gamma = \{[1, 3]\} \cup \{[3, 5], [4, 9]\}$
- $\Gamma = \{[1, 3]\} \cup \{[3, 5], [4, 7], [7, 9]\}$
- $\Gamma = \{[1, 3]\} \cup \{[3, 5], [4, 8], [7, 9]\}$
- $\Gamma = \{[1, 3]\} \cup \{[3, 7], [7, 9]\}$
- $\Gamma = \{[1, 3]\} \cup \{[3, 7], [6, 9]\}$
- $\Gamma = \{[1, 3]\} \cup \{[3, 8], [7, 9]\}$.

Then corresponding to the prime sequence $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_r$ of Δ we associate the ideal P_Γ in the polynomial ring $K[X]$ as

$$P_\Gamma = P_{\Gamma_1} + \dots + P_{\Gamma_r},$$

where P_{Γ_i} is the prime ideal associated to Γ_i in Definition 2.4.

Theorem 2.6. Let Δ be a union of block adjacent simplicial complexes as Definition 2.4. Then the minimal primary decomposition of J_Δ is given by

$$J_\Delta = \bigcap_{\Gamma \in \mathcal{A}_\Delta} P_\Gamma.$$

Proof. We have $\mathcal{V}(J_\Delta) = \mathcal{V}(J_{\Delta_1}) \cap \dots \cap \mathcal{V}(J_{\Delta_r})$. Therefore Theorem 2.3 implies that corresponding to each matrix X of $\mathcal{V}(J_\Delta)$ there exist prime sequences Γ_i associated to Δ_i such that $X \in \mathcal{V}(P_{\Gamma_i})$ for all i , and so $X \in \mathcal{V}(P_{\Gamma_1} + \dots + P_{\Gamma_r}) = \mathcal{V}(P_\Gamma)$ in which $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_r$. One can observe that Γ is a prime sequence of Δ . On the other hand, $J_{\Delta_i} \subseteq P_{\Gamma_i}$ implies that $J_\Delta \subseteq P_\Gamma$ for all prime sequences Γ . The primality of P_Γ follows by Theorem 2.8. The minimality of the given primary decomposition of J_Δ holds by the same argument given in the proof of [22, Corollary 4.6]. \square

Example 2.7. Let Δ be a simplicial complex with $\mathcal{F}(\Delta) = \{123, 134, 124, 234, 456, 567\}$. Then one can observe that $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1 is the full 2-skeleton of the simplex on the vertex set $\{1, 2, 3, 4\}$, and Δ_2 is the adjacent simplicial complex on the vertex set $\{4, 5, 6, 7\}$ with $\mathcal{F}(\Delta_2) = \{456, 567\}$. Then $J_\Delta = P_1 \cap P_2$ is the minimal prime decomposition of J_Δ , where

- $P_1 = ([123], [134], [124], [234], x_{16}, x_{26}, x_{36})$
- $P_2 = ([123], [134], [124], [234], [456], [457], [467], [567])$.

Theorem 2.8. *Let Δ be a union of block adjacent simplicial complexes, and Γ be a prime sequence of Δ . Then the ideal P_Γ is prime.*

Proof. We do induction on r . The case $r = 1$ holds by Theorem 2.3. Assume that $r > 1$. First notice that the vertex v_1 does not take the position $m - 1$ in any facet of Δ , and the vertex $v_1 - k \geq v_2$ does not take the position $m - 1 - k$ in any facet of Δ . Assume that $v_1 - \beta \leq v_1$ be the last index such that $[a_1 \dots a_t | b_1 \dots b_{t-1}, v_1 - \beta]$ belongs to the generating set of P_{Γ_1} for some indices $a_1 < \dots < a_t$ and $b_1 < \dots < b_{t-1} < v_1 - \beta$. Therefore the variable $y_1 = x_{m-1-\beta, v_1-\beta}$ does not appear in the support of the generators of $\text{in}_<(P_\Gamma)$ which implies that y_1 is a regular element modulo P_Γ . We write

$$\begin{aligned} P_\Gamma &= P_{\Gamma_1} + P_{\Gamma_2} + \sum_{i=3}^r P_{\Gamma_i} \\ &= (P_{11} + P_{12} + P_1) + (Q_{11} + \dots + Q_{1w_1} + Q_1) + \sum_{i=3}^r P_{\Gamma_i}, \end{aligned}$$

in which for each minor $[a_1 \dots a_t | b_1 \dots b_t]$ in the generating set of $P_1 + Q_1$ we have $b_\ell \neq v_1 - \beta$ for all ℓ . Now, applying Lemma 2.11 together with Lemma 2.10 for $s = 1$ and $q = v_1 - \beta$, we get $(R/P_\Gamma)_{y_1} \cong (R/L_1 + L_2)_{y_1}$, where $L_1 = P'_{11} + P'_{12} + P_1$ and $L_2 = Q'_{11} + \dots + Q'_{1w_1} + Q_1 + \sum_{i=3}^r P_{\Gamma_i}$. Applying the same method for variables $y_1 = x_{m-1-\beta, v_1-\beta}, y_2 = x_{m-1-\beta-1, v_1-\beta-1}, \dots, y_s = x_{m+u_2-v_1-1, u_2}$ consequently, we get

$$(R/P_\Gamma)_{y_1 \dots y_s} \cong (R/I_1 + I_2)_{y_1 \dots y_s},$$

for ideals I_1 and I_2 with $(R/P_{\Gamma_1})_{y_1 \dots y_s} \cong (R/I_1)_{y_1 \dots y_s}$ and $(R/P_{\Gamma_2 \cup \dots \cup \Gamma_r})_{y_1 \dots y_s} \cong (R/I_2)_{y_1 \dots y_s}$. By induction hypothesis I_1 and I_2 are prime ideals. On the other hand I_1 and I_2 live on different polynomial rings with no common variable which implies that $I_1 + I_2$ is a prime ideal. The fact that all variables y_1, \dots, y_s are chosen to be nonzerodivisor in each step, implies that P_Γ is indeed a prime ideal. \square

Theorems 2.6 and 2.8 imply the following corollary.

Corollary 2.9. *Let $\Delta = \Delta_1 \cup \dots \cup \Delta_r$ be a union of block adjacent simplicial complexes. Then $J_{\Delta_1} + \dots + J_{\Delta_r}$ is a prime splitting of J_Δ .*

2.3. Localization lemmata. The proof of the primality of the associated ideals to the prime sequences constructed in Definition 2.4 is based on finding nonzerodivisor elements modulo the ideal. Then we will use localization argument with respect to these nonzerodivisor elements. The procedure of localization often follows by next lemmata.

Lemma 2.10. *Let K be a field, X be an $m \times n$ -matrix of indeterminates and $I \subset S = K[X]$ an ideal generated by a set \mathcal{G} of minors. Furthermore, let x_{ij} be an entry of X . We assume that for each minor $[a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$ with $i \notin \{a_1, \dots, a_t\}$ and $j \notin \{b_1, \dots, b_t\}$, the minors $[ia_1 \dots \hat{a}_k \dots a_t | b_1 \dots b_t] \in \mathcal{G}$, or the minors $[a_1 \dots a_k \dots a_t | jb_1 \dots \hat{b}_k b_t] \in \mathcal{G}$ for all k . Then $(R/I)_{x_{ij}} \cong (R/J)_{x_{ij}}$ where J is generated by*

- the minors $[a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$ with $a_\ell \neq i$ for all ℓ or $b_k \neq j$ for all k ,*
- the minors $[a_1 \dots \hat{a}_\ell \dots a_t | b_1 \dots \hat{b}_k \dots b_t]$ where $[a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$ and $a_\ell = i$ and $b_k = j$ for some ℓ and k .*

Proof. For simplicity we may assume that $i = 1$ and $j = 1$. We apply the automorphism $\varphi: S_{x_{11}} \rightarrow S_{x_{11}}$ with

$$x_{ij} \mapsto x'_{ij} = \begin{cases} x_{ij} + x_{i1}x_{11}^{-1}x_{1j}, & \text{if } i \neq 1 \text{ and } j \neq 1, \\ x_{ij}, & \text{if } i = 1 \text{ or } j = 1. \end{cases}$$

Let $I' \subset S_{x_{11}}$ be the ideal which is the image of $IS_{x_{11}}$ under the automorphism φ . Then $(R/I)_{x_{11}} \cong S_{x_{11}}/I'$. The ideal I' is generated in $S_{x_{11}}$ by the elements $\varphi(\mu_M)$ where $\mu_M \in \mathcal{G}$. Note that if $\mu_M = [a_1 \dots a_t | b_1 \dots b_t]$, then $\varphi(\mu_M) = \det(x'_{a_i b_j})_{i,j=1,\dots,t}$.

In the following we may assume that $a_1 < a_2 < \dots < a_t$ and $b_1 < b_2 < \dots < b_t$ for $\mu_M = [a_1 \dots a_t | b_1 \dots b_t] \in \mathcal{G}$. Let us first consider the case that $a_1 = 1$. If $b_1 \neq 1$, then $\varphi(\mu_M)$ is the determinant of the matrix

$$\begin{pmatrix} x_{1b_1} & x_{1b_2} & \dots & x_{1b_t} \\ x_{a_2b_1} + x_{a_21}x_{11}^{-1}x_{1b_1} & x_{a_2b_2} + x_{a_21}x_{11}^{-1}x_{1b_2} & \dots & x_{a_2b_t} + x_{a_21}x_{11}^{-1}x_{1b_t} \\ \vdots & \vdots & \dots & \vdots \\ x_{a_tb_1} + x_{a_t1}x_{11}^{-1}x_{1b_1} & x_{a_tb_2} + x_{a_t1}x_{11}^{-1}x_{1b_2} & \dots & x_{a_tb_t} + x_{a_t1}x_{11}^{-1}x_{1b_t} \end{pmatrix}$$

By subtracting suitable multiples of the first row from the other rows we see that

$$\varphi(\mu_M) = \det(x_{a_i b_j})_{i,j=1,\dots,t} = \mu_M.$$

In the case that $b_1 = 1$, the element $\varphi(\mu_M)$ is the determinant of the matrix

$$\begin{pmatrix} x_{11} & x_{1b_2} & \dots & x_{1b_t} \\ x_{a_21} & x_{a_2b_2} + x_{a_21}x_{11}^{-1}x_{1b_2} & \dots & x_{a_2b_t} + x_{a_21}x_{11}^{-1}x_{1b_t} \\ \vdots & \vdots & \dots & \vdots \\ x_{a_t1} & x_{a_tb_2} + x_{a_t1}x_{11}^{-1}x_{1b_2} & \dots & x_{a_tb_t} + x_{a_t1}x_{11}^{-1}x_{1b_t} \end{pmatrix}$$

Applying suitable row operations we obtain the matrix

$$\begin{pmatrix} 1 & x_{11}^{-1}x_{1b_2} & \dots & x_{11}^{-1}x_{1b_t} \\ 0 & x_{a_2b_2} & \dots & x_{a_2b_t} \\ \vdots & \vdots & \dots & \vdots \\ 0 & x_{a_tb_2} & \dots & x_{a_tb_t} \end{pmatrix}$$

It follows that $\varphi(\mu_M) = \det(x_{a_i b_j})_{i,j=2,\dots,t}$.

Now we consider the case that $a_1 > 1$. If $b_1 = 1$, then by subtracting suitable multiples of the first column from the other columns we see that

$$\varphi(\mu_M) = \det(x_{a_i b_j})_{i,j=1,\dots,t} = \mu_M.$$

Now assume that $b_1 > 1$. By our assumption on μ_M we have that

$$[ia_1 \dots \hat{a}_k \dots a_t | b_1 \dots b_t] \in \mathcal{G} \quad \text{or} \quad [a_1 \dots a_k \dots a_t | jb_1 \dots \hat{b}_k \dots b_t] \in \mathcal{G} \quad \text{for all } k.$$

Assume that $[ia_1 \dots \hat{a}_k \dots a_t | b_1 \dots b_t] \in \mathcal{G}$ for all k . Then applying suitable row operations we obtain $[1a_1a_2 \dots a_t | 1b_1b_2 \dots b_t]$ is equal to the determinant of the matrix

$$\begin{pmatrix} x_{11} & x_{1b_1} & \dots & x_{1b_t} \\ 2x_{a_11} & x_{a_1b_1} + x_{a_11}x_{11}^{-1}x_{1b_1} & \dots & x_{a_1b_t} + x_{a_11}x_{11}^{-1}x_{1b_t} \\ \vdots & \vdots & \dots & \vdots \\ 2x_{a_t1} & x_{a_tb_1} + x_{a_t1}x_{11}^{-1}x_{1b_1} & \dots & x_{a_tb_t} + x_{a_t1}x_{11}^{-1}x_{1b_t} \end{pmatrix}$$

Now comparing the expansion of the above matrix by the first column with the expansion of $[1a_1a_2 \dots a_t | 1b_1 \dots b_t]$ by the first column, we see that

$$\varphi(\mu_M) = \mu_M + \sum_{i=1}^t (-1)^{i-1} 2x_{a_i 1} x_{11}^{-1} \varphi(\gamma_i) - \sum_{i=1}^t (-1)^{i-1} x_{a_i 1} x_{11}^{-1} \gamma_i,$$

where $\gamma_i = [1a_1 \dots \hat{a}_i \dots a_t | b_1 \dots b_t]$ for all i . We have already shown that $\varphi(\gamma_i) = \gamma_i \in I'$ for all i which implies that

$$\varphi(\mu_M) = \mu_M + \sum_{i=1}^t (-1)^{i-1} x_{11}^{-1} x_{a_i 1} \gamma_i,$$

and we conclude that $\mu_M \in I'$. Similar argument works for the case that all minors $[a_1 \dots a_k \dots a_t | j b_1 \dots b_t] \in \mathcal{G}$. These calculations show that $I' = JS_{x_{11}}$, as desired. \square

The following technical lemma is a consequence of applying Lemma 1.8 and Lemma 2.10 for particular ideals from the proof of Theorem 2.8.

Lemma 2.11. *Let $P = P_1 + \sum_{\ell=1}^s (P_{\ell 1} + P_{\ell 2}) + \sum_{\ell=1}^{w_\ell} (Q_{\ell 1} + \dots + Q_{\ell w_\ell})$, where*

- P_1 is generated by a collection of minors such that for each minor $[a_1 \dots a_t | b_1 \dots b_t]$ in the generating set of P_1 we have $b_\ell \neq q$ for all ℓ ;
- $P_{\ell k}$ is generated by maximal minors of the submatrix $Y_{\ell k} = X_{S_\ell}[p_{\ell k}, q]$ for all ℓ and k , where $S_\ell = \{1, 2, \dots, m-s, m-\ell+1, \dots, m\}$ and $p_{11} \leq p_{12} \leq \dots \leq p_{s1} \leq p_{s2} < q$;
- $Q_{\ell k}$ is generated by maximal minors of the submatrix $X_{\ell k} = X_{T_\ell}[B_{\ell k}]$ for all ℓ and k , where $B_{\ell k} = \{i_{\ell k}, \dots, q+s-\ell, q+s, \dots, j_{\ell k}\}$, $T_\ell = \{1, 2, \dots, m-s, m-s+\ell, \dots, m\}$, and $i_{11} \leq i_{12} \leq \dots \leq i_{1w_1} \leq \dots \leq i_{sw_s} \leq q$ and $j_{11} \leq j_{12} \leq \dots \leq j_{sw_s}$;
- generators of P form a Gröbner basis with respect to $<_{\text{lex}}$, and the variable $x_{m-s,q}$ does not appear in the support of any generator of P .

Then $(R/P)_{x_{m-s,q}} = (R/P')_{x_{m-s,q}}$, where

$$P' = P_1 + \sum_{\ell=1}^s (P'_{\ell 1} + P'_{\ell 2}) + \sum_{\ell=1}^{w_\ell} (Q'_{\ell 1} + \dots + Q'_{\ell w_\ell}).$$

Here $P'_{\ell k}$ is generated by maximal minors of the submatrix $Y'_{\ell k} = X_{S_\ell \setminus \{m-s\}}^{[p_{\ell k}, q-1]}$ for all ℓ and k , and $Q'_{\ell k}$ is generated by maximal minors of the submatrix $X'_{\ell k} = X_{T'_\ell}[B'_{\ell k}]$, where $T'_\ell = T_\ell \setminus \{m-s\}$ and $B'_{\ell k} = B_{\ell k} \setminus \{q\}$. Moreover the generators of P' form a Gröbner basis with respect to $<_{\text{lex}}$, and the variable $x_{m-s-1,q-1}$ does not appear in the support of any element in the generating set of P' .

Proof. One can observe that the variable $x_{m-s,q}$ does not appear among initial terms of the generators of P . Since the generators of P form a Gröbner basis with respect to $<_{\text{lex}}$, we deduce that $x_{m-s,q}$ is nonzerodivisor modulo P . Now applying Lemma 2.10 for the variable $x_{m-s,q}$ we get $(R/P)_{x_{m-s,q}} = (R/P')_{x_{m-s,q}}$. In order to show that the generators of P' form a Gröbner basis for P' with respect to $<_{\text{lex}}$, for every pair f and g of minors from different submatrices, we will find an element h in P' with $\text{in}(h) = \text{LCM}(\text{in}(f), \text{in}(g))$. Assume that f is a minor of the matrix $X'_{\ell k}$, and g is a minor of the matrix $X'_{\ell' k'}$, where $\ell' \leq \ell$ and $k' \leq k$. Then the polynomial h is given by Lemma 1.8 (ii), since $[p_{\ell k}, q-1] \subseteq [p_{\ell' k'}, q-1]$ and $S_\ell \setminus \{m-s\} \subseteq S_{\ell'} \setminus \{m-s\}$. Similarly in the case that the set of columns of $X'_{\ell k}$ is a subset of the set of columns of $X'_{\ell' k'}$ we deduce that the set of rows of $X'_{\ell k}$ is the subset of the set of rows of $X'_{\ell' k'}$, and so the corresponding polynomial h is given by

Lemma 1.8. Otherwise initial terms of f and g are relatively prime which implies that for the polynomial $h = fg$ we have $\text{in}(h) = \text{in}(f)\text{in}(g)$. Therefore the generators of P' form a Gröbner basis for P' with respect to $<_{\text{lex}}$. \square

3. DETERMINANTAL FACET IDEALS OF GENERALIZED FORESTS

Here assume that $\Delta = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_r$ is a closed simplicial complex with

- (a) $|V(\Delta_i) \cap V(\Delta_j) \cap V(\Delta_k)| = 0$ for all $i < j < k$,
- (b) $|V(\Delta_i) \cap V(\Delta_j)| \leq 1$ for all $i < j$.

We consider the simple graph G_Δ whose vertices $1, \dots, r$ are identified with complexes $\Delta_1, \Delta_2, \dots, \Delta_r$ and two vertices i and j are adjacent if and only if $V(\Delta_i) \cap V(\Delta_j) \neq \emptyset$. Then we study the minimal primes of J_Δ , where the associated graph is a tree. Two different types of complexes have been studied here. In the first subsection the components Δ_i are unions of block adjacent complexes, and in the second part, components Δ_i are full skeletons of some simplices. Here we will assume that $\dim(F) > 1$ for each facet F of Δ .

3.1. Forests whose vertices identified by block adjacent complexes. Assume that the components Δ_i of Δ are complexes on the vertex set $[u_i, v_i]$ as Definition 2.4 such that

- (c) $V(\Delta_i) \cap V(\Delta_j) \subseteq \{u_i + t\} \cap [v_j + t - (m - 3), v_j]$ or
 $V(\Delta_i) \cap V(\Delta_j) \subseteq \{v_i - t\} \cap [u_j, u_j + m - t - 3]$ for some $0 \leq t \leq m - 3$.

Lemma 3.1. *Let $\ell \in \{u_i + t\} \cap [v_j + (m - 3) - t, v_j]$ or $\ell \in \{v_i - t\} \cap [u_j, u_j + m - t - 3]$ for some t . Then there exists k such that $x_{k\ell}$ does not appear in the supports of the initial terms of the generators of J_Δ .*

Proof. One can observe that $u_i + t$ can just take positions $1, 2, \dots, t + 1$ in facets of the simplicial complex Δ_i , and $v_i - t$ can take only positions $m - t, \dots, m$ in facets of Δ_i . Therefore for $\ell \in \{u_i + t\} \cap [v_j - (m - 3 - t), v_j]$, the variable $x_{t+2,\ell}$ does not appear in the supports of initial terms of the generators of J_Δ , and in the case that $\ell \in \{v_i - t\} \cap [u_j, u_j + m - t - 3]$, the variable $x_{m-t-1,\ell}$ does not appear in the supports of the initial terms of generators of J_Δ . \square

Theorem 3.2. *If G_Δ is a tree, then the primary decomposition of J_Δ is given by*

$$J_\Delta = \bigcap_{\Gamma \in \mathcal{A}} P_\Gamma,$$

where \mathcal{A} consists sequences $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_r$ in which $\Gamma_i \in \mathcal{A}_{\Delta_i}$ and $P_\Gamma = P_{\Gamma_1} + \dots + P_{\Gamma_r}$.

Proof. Let $\Delta = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_r$, where Δ_i is a block adjacent simplicial complex. Note that by Theorem 1.5 the ideal J_Δ is radical and it can be written as the intersection of its prime ideals. First we show that the generators of P_Γ form a Gröbner basis with respect to $<_{\text{lex}}$. Theorem 1.10 implies that the generators of P_{Γ_i} form a Gröbner basis for P_{Γ_i} for all i . The intersection properties (a), (b) and (c) guarantee that the initial terms of the generators of P_{Γ_i} and P_{Γ_j} do not have any common variable. Therefore for every pair of the generators f of P_{Γ_i} and g of P_{Γ_j} , their initial terms are relatively prime and $\text{in}(fg) = \text{in}(f)\text{in}(g)$ which implies that the generators of J_Δ form a Gröbner basis for J_Δ .

The proof of the primality of ideals P_Γ is by induction on r , i.e. the number of components of Δ . Since G_Δ is a tree, without loss of generality we may assume that Δ_1 intersects with just one adjacent simplicial complex, say Δ_2 in the vertex $u_1 + t$ for some $0 \leq t \leq m - 3$. Lemma 3.1 shows that $x_{t+2,\ell}$ does not appear among initial terms of the generators of P_Γ which implies that the variable $x_{t+2,\ell}$ is a nonzerodivisor modulo P_Γ , since

the generators of P_Γ form a Gröbner basis for P_Γ . Now by applying Lemma 2.10 for the variable $x_{t+2,\ell}$ we get $(R/P_\Gamma)_{x_{t+2,\ell}} \cong (R/L)_{x_{t+2,\ell}}$, where $L = L_1 + L_2 + \sum_{i=3}^r P_{\Gamma_i}$ such that L_1 and L_2 do not have any minor involving the ℓ^{th} column. On the other hand, L_1 lives in the polynomial ring on variables x_{ij} 's, where $i \in \{1, \dots, t+1, t+3, \dots, m\}, j \in V(\Delta_1) \setminus \{\ell\}$, and these variables do not appear in the ideal $L_2 + \sum_{i=3}^r P_{\Gamma_i}$. Therefore P_Γ is prime if and only if L_1 and L_2 are prime ideals. Since initial terms of the generators of P_{Γ_1} and $P_{\Gamma_2} + \dots + P_{\Gamma_r}$ are polynomials in different variables than $x_{t+2,\ell}$, it follows that $x_{t+2,\ell}$ is regular modulo both ideals. Then similar to the above argument we have

$$(R/P_{\Gamma_1})_{x_{t+2,\ell}} \cong (R/L_1)_{x_{t+2,\ell}} \quad \text{and} \quad (R/\sum_{i=2}^r P_{\Gamma_i})_{x_{t+2,\ell}} \cong (R/L_2 + \sum_{i=3}^r P_{\Gamma_i})_{x_{t+2,\ell}}$$

which are integral domains by induction hypothesis. Therefore L_1 and $L_2 + P_{\Gamma_3} + \dots + P_{\Gamma_r}$ are prime ideals, as desired. Now note that

$$\mathcal{V}(J_\Delta) = \mathcal{V}(J_{\Delta_1}) \cap \dots \cap \mathcal{V}(J_{\Delta_r}).$$

Therefore corresponding to each matrix X of $\mathcal{V}(J_\Delta)$ there exist prime sequences Γ_i associated to Δ_i such that $X \in \mathcal{V}(P_{\Gamma_i})$ for all i . Therefore $X \in \mathcal{V}(P_{\Gamma_1} + \dots + P_{\Gamma_r}) = \mathcal{V}(P_\Gamma)$. On the other hand, $J_{\Delta_i} \subseteq P_{\Gamma_i}$ for all i which implies that $J_\Delta \subseteq P_\Gamma$, as desired. The minimality of the primary decomposition of J_Δ follows by the proof of [22, Corollary 4.6]. \square

Example 3.3. Assume that $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$ is a 3-dimensional simplicial complex on [11], where Δ_1 is the adjacent simplicial complex on the vertex set $\{1, 2, 3, 4, 5\}$ with $\mathcal{F}(\Delta_1) = \{1234, 2345\}$, Δ_2 is the adjacent simplicial complex on the vertex set $\{4, 6, 7, 8, 9\}$ with $\mathcal{F}(\Delta_2) = \{4678, 6789\}$, and Δ_3 is the simplex on the vertex set $\{5, 9, 10, 11\}$. One can observe that G_Δ is a path on three vertices. By Theorem 2.6 we have that the prime ideals of J_{Δ_1} are the ideals J_Γ and P_1 , where J_Γ is the ideal associated to the 3-skeleton of the simplex Γ on the vertex set $\{5\}$, and $P_1 = ([123|234], [124|234], [234|234], [134|234])$. Also the prime ideals of J_{Δ_2} are the ideals $J_{\Gamma'}$ associated to the 3-skeleton of the simplex Γ' on the vertex set $\{4, 6, 7, 8, 9\}$, and the ideal $P_2 = ([123|678], [124|678], [234|678], [134|678])$. Therefore by Theorem 3.2 minimal prime ideals of J_Δ are

- $J_\Gamma + J_{\Gamma'} + J_{\Delta_3}$
- $J_\Gamma + P_2 + J_{\Delta_3}$
- $P_1 + J_{\Gamma'} + J_{\Delta_3}$
- $P_1 + P_2 + J_{\Delta_3}$.

Corollary 3.4. *If G_Δ is a tree, then $J_{\Delta_1} + \dots + J_{\Delta_r}$ is a prime splitting of J_Δ .*

3.2. Forests whose vertices identified by clique complexes. Here we study the determinantal facet ideal of Δ in which all components are full skeletons of some simplices, i.e. $\Delta = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_r$ is a clique decomposition of Δ , and we assume that H_Δ is the corresponding graph.

Theorem 3.5. *If H_Δ is a tree, then J_Δ is a prime ideal.*

In order to prove the primality of J_Δ we need to find some nonzerodivisor elements modulo J_Δ which are given by the following lemma. This Lemma has been stated in [12] for pure simplicial complexes, and the same proof works for non-pure complexes, as well.

Lemma 3.6. *Let Δ be a closed $(m-1)$ -dimensional simplicial complex with the property that any m pairwise distinct cliques of Δ have an empty intersection. Assume further that K is infinite. Then each variable x_{ij} is regular modulo J_Δ .*

Proof. We consider the ideal $I = J_\Delta + (x_{1j}, \dots, x_{mj})$ which equals to $J_{\Delta'} + (x_{1j}, \dots, x_{mj})$ in which Δ' is the simplicial complex whose facets are those of Δ not containing the vertex j . Assume that $j \in V(\Delta_i)$ for $i = 1, \dots, s$. Then $\Delta' = \Delta'_1 \cup \dots \cup \Delta'_s$ is again a closed simplicial complex. By Corollary 1.7 we have

$$\begin{aligned} \text{height } I &= \text{height } J_{\Delta'} + m = \sum_{i=1}^s (|\Delta_i| - 1 - \dim(\Delta_i)) + \sum_{i=s+1}^r (|\Delta_i| - \dim(\Delta_i)) + m \\ &= \text{height } J_\Delta - s + m > \text{height } J_\Delta. \end{aligned}$$

Our considerations show that $I/J_\Delta \subset R/J_\Delta$ has positive height. Since R/J_Δ is Cohen–Macaulay and K is infinite, it follows that a generic linear combination $a_1x_{1j} + a_2x_{2j} + \dots + a_mx_{mj}$ of variables x_{1j}, \dots, x_{mj} (whose residue classes generate I/J_Δ) is regular modulo J_Δ . Since the above linear combination is generic, we may assume that $a_i = 1$.

Now we consider the linear automorphism $\varphi: S \rightarrow S$ with $\varphi(x_{ik}) = a_1x_{1k} + a_2x_{2k} + \dots + a_mx_{mk}$ for $k = 1, \dots, n$ and $\varphi(x_{\ell k}) = x_{\ell k}$ for $\ell \neq i$ and all k . Let X' be the matrix whose entries are elements $\varphi(x_{\ell k})$ for $\ell = 1, \dots, m$ and $k = 1, \dots, n$. Then X' is obtained from X by elementary row operations. It follows that $\varphi(J_\Delta) = J_\Delta$. By our choice of φ we have that $y_{ij} = \varphi(x_{ij})$ is regular modulo J_Δ . Since $J_\Delta = \varphi(J_\Delta)$, it follows that $x_{ij} = \varphi^{-1}(y_{ij})$ is regular modulo $\varphi^{-1}(J_\Delta) = \varphi^{-1}(\varphi(J_\Delta)) = J_\Delta$, as desired. \square

Proof of Theorem 3.5. Lemma 3.6 enables us to generalize some results of [12] to cover the case of non-pure simplicial complexes. In fact Lemma 3.6 guarantees that we can apply the same techniques given in [12] for primality of facet ideals of pure simplicial complexes, in order to prove the primality of *non-pure* closed simplicial complexes. The proof of primality of J_Δ follows word by word from the proof given in [12, Theorem 3.2 (a)] for the *pure* case. \square

Example 3.7. Assume that $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$ is a 3-dimensional simplicial complex on [9], where Δ_1 is the simplex on the vertex set $\{1, 2, 3, 4\}$ with $\mathcal{F}(\Delta_1) = \{1234\}$, Δ_2 is the full 2-skeleton of the simplex on the vertex set $\{2, 5, 6, 7\}$ with $\mathcal{F}(\Delta_2) = \{256, 267, 257, 567\}$, and Δ_3 is the simplex on the vertex set $\{4, 8, 9\}$. One can observe that H_Δ is a path on three vertices and Δ satisfies the conditions of Theorem 3.5 which implies that J_Δ is a prime ideal. Note that the condition (c) does not hold for Δ .

3.3. Further questions and Future Works. A natural question to ask is whether our techniques can be used to obtain minimal primes of a more general class of mixed determinantal ideals. It turns out that similar techniques and ideas of the proof of Theorem 3.2 work for the ideals whose graphs are cycles, and cactus graphs, i.e., connected graphs in which each edge belongs to at most one cycle. Here we have removed the proofs, and we just mention an example in which G_Δ is a cactus graph.

Example 3.8. Let $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, where Δ_1 is the adjacent simplicial complex on the vertex set $\{1, 2, 3, 4\}$, Δ_2 is the adjacent 2-dimensional complex on the vertex set $\{4, 5, 6, 7\}$, Δ_3 is the adjacent 2-dimensional complex on the vertex set $\{3, 7, 8, 9\}$ and Δ_4 is the 2-skeleton of the simplex on the vertex set $\{9, 10, 11, 12\}$. Let G_Δ be the graph associated to Δ on the vertex set [4] with edges $\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 4\}$ which is a cactus graph. It is straightforward to check that Δ is closed. As we know J_{Δ_4} is a prime ideal, and by Theorem 2.3 we already know the minimal primary decomposition of J_{Δ_1} , J_{Δ_2} and J_{Δ_3} which are given as

- $J_{\Delta_1} = ([12|23], [13|23], [23|23]) \cap ([123], [124], [134], [234]) = P_1 \cap P_2,$
- $J_{\Delta_2} = ([12|56], [13|56], [23|56]) \cap ([456], [457], [467], [567]) = Q_1 \cap Q_2,$

- $J_{\Delta_3} = ([12|78], [13|78], [23|78]) \cap ([378], [379], [389], [789]) = L_1 \cap L_2$.

Then the minimal primes of J_Δ are

$$J_{\Delta_4} + P_i + Q_j + L_k \quad \text{for } i = 1, 2, j = 1, 2, k = 1, 2.$$

Another direction in order to generalize the results is as follows: given two pure $(k-1)$ -dimensional simplicial complexes on the vertex set $[n]$, we define the ideal

$$J_{\Delta_1, \Delta_2} = ([a_1 \dots a_k | b_1 \dots b_k] : \{a_1, \dots, a_k\} \in \mathcal{F}(\Delta_1) \text{ and } \{b_1, \dots, b_k\} \in \mathcal{F}(\Delta_2))$$

which is called the *determinantal facet ideal* associated to the pair (Δ_1, Δ_2) . By our computations using the software Singular [15] we note that J_{Δ_1, Δ_2} is a radical ideal just in the case that either Δ_1 or Δ_2 is a full skeleton of a simplex. We are interested to see how algebraic properties of J_{Δ_1} and J_{Δ_2} influence algebraic properties of J_{Δ_1, Δ_2} . We note that V. Ene et al. in [13] recently studied the case in which Δ_1 and Δ_2 are both graphs. Is there any technic to express minimal primes of J_{Δ_1, Δ_2} in terms of minimal primes of determinantal facet ideals of J_{Δ_1} and J_{Δ_2} ? Our examples show that there exist ideals J_{Δ_1, Δ_2} with nice decompositions into smaller ideals $J_{\Gamma_1}, \dots, J_{\Gamma_t}$ such that minimal primes of J_{Δ_1, Δ_2} can be determined from minimal primes of J_{Γ_i} . It is interesting to determine classes of ideals with this property. Another approach could be to get information on associated primes of these ideals.

Remark 3.9. Assume that Δ is an $(m-1)$ -dimensional pure simplicial complex on the vertex set $[n]$. By [12, Theorem 1.1] we have that the generators of J_Δ form a Gröbner basis with respect to the lexicographical order induced by $x_{11} > \dots > x_{1n} > \dots > x_{mn}$ if and only if Δ is a closed simplicial complex. Hence we conclude that the generators of J_Δ form a Gröbner basis with respect to the lexicographical order induced by every term order on variables x_{ij} , if and only if Δ is a full skeleton of a simplex. Therefore by a result of Bernstein and Zelevinsky from [1], we have that the generators of J_Δ form a universal Gröbner basis of J_Δ , i.e., a Gröbner basis with respect to every term order on $K[X]$, if and only if Δ is the $(m-1)$ -skeleton of the simplex on $[n]$, i.e. J_Δ is the ideal generated by all maximal minors of X .

A natural question to ask is whether the same result holds for non-pure simplicial complexes. We expect that the generators of a mixed determinantal facet ideal of J_Δ form a universal Gröbner basis for J_Δ if and only if Δ is a disjoint union of simplices.

4. MINIMAL FREE RESOLUTION OF DETERMINANTAL FACET IDEALS

In this section we study the minimal free resolution of a determinantal facet ideal for a closed simplicial complex Δ with clique decomposition $\Delta = \Delta_1 \cup \dots \cup \Delta_r$. We construct the minimal free resolution of J_Δ as a tensor product of the minimal free resolution of facet ideals of its cliques. The minimal free resolution of the determinantal ideal I generated by all maximal minors of the generic $m \times n$ matrix X , is given by the Eagon-Northcott complex. Here we state the following known result from [10].

Proposition 4.1. *Let $X = (x_{ij})$ be an $m \times n$ matrix and I be the ideal generated by all maximal minors of X . Then the minimal free resolution of R/I is given by the Eagon-Northcott complex, where the k^{th} module in the resolution is $D_k(S^m) \otimes \wedge^{k+m}(S^n)$ and the first map of the complex consists of the elements of $G(J_\Delta)$ i.e. the generating set of J_Δ , and whose later maps are of the form*

$$\partial_k : D_k(S^m) \otimes \wedge^{k+m}(S^n) \rightarrow D_{k-1}(S^m) \otimes \wedge^{k+m-1}(S^n)$$

Here D_k is the divided power algebra and matrices are chosen with respect to natural basis elements $e_{i_1} \dots e_{i_k} \otimes g_{j_1} \wedge g_{j_2} \wedge \dots \wedge g_{j_{k+m}}$ for $i_1 \leq i_2 \leq \dots \leq i_k$ and $j_1 < \dots < j_{k+m}$, where e_1, \dots, e_m are denoted for basis elements of rows of X , and g_1, \dots, g_n for basis elements of columns of X .

Remark 4.2. The Eagon-Northcott complex associated to R/I is a minimal *linear* free resolution of R/I , and the Betti numbers are given by

$$\beta_i(R/I) = \binom{n}{m+i} \binom{m+i-1}{i}.$$

Remark 4.3. We recall that, for a monomial ideal $I \subset R = K[x_1, \dots, x_n]$ with minimal generators m_1, \dots, m_k , the set of monomials generating the quotient ideal $\langle m_1, \dots, m_{i-1} \rangle : m_i$ is denoted by $\text{set}(m_i)$, i.e.,

$$\text{set}(m_i) = \{u : u \in \langle m_1, \dots, m_{i-1} \rangle : m_i\}.$$

We say that I has linear quotients with respect to the order m_1, \dots, m_k of its generators if $\text{set}(m_i) \subseteq \{x_1, \dots, x_m\}$. In this case the Betti numbers of I are given by

$$\beta_i(R/I) = \sum_{u \in G(I)} \binom{|\text{set}(u)|}{i},$$

where $G(I)$ is the minimal generating set of I , see [19].

Remark 4.4.

$$\sum_{k=0}^n \binom{k}{i} \binom{k+a}{a} = \binom{n+a+1}{i+a+1} \binom{i+a}{i}$$

We recall that by $<_{\text{lex}}$, we always mean the lexicographical order induced by the natural order of variables $x_{11} > x_{12} > \dots > x_{1n} > \dots > x_{mn}$.

Lemma 4.5. Assume that I is the ideal generated by maximal minors of X . Then

$$\beta_i(R/\text{in}_{<} I) = \beta_i(R/I) \quad \text{for all } i.$$

Proof. We have $\text{in}_{<} I = \langle x_{1i_1} x_{2i_2} \dots x_{mi_m} : 1 \leq i_1 < i_2 < \dots < i_m \leq n \rangle$. It's easy to check that I has linear quotients with respect to the order $m_1 >_{\text{lex}} m_2 >_{\text{lex}} \dots >_{\text{lex}} m_t$ on its generators, and for $u = x_{1i_1} x_{2i_2} \dots x_{mi_m}$ we have

$$\text{set}(u) = \{x_{11}, x_{12}, \dots, x_{1,i_1-1}\} \cup \{x_{2,i_1+1}, \dots, x_{2,i_2-1}\} \cup \dots \cup \{x_{m,i_{m-1}+1}, \dots, x_{m,i_m-1}\}.$$

Therefore $|\text{set}(u)| = (i_1 - 1) + (i_2 - i_1 - 1) + \dots + (i_m - i_{m-1} - 1) = i_m - m$. One can see that the number of monomials $x_{1\ell_1} x_{2\ell_2} \dots x_{m-1\ell_{m-1}} x_{mi_m}$ divisible by x_{mi_m} in the generating set of $\text{in}_{<} I$, are exactly $\binom{i_m-1}{m-1}$ which implies that

$$\beta_i(R/\text{in}_{<} I) = \sum_{i_m=m}^n \binom{i_m-m}{i} \binom{i_m-1}{m-1} \quad \text{for all } i.$$

Now Remarks 4.2 and 4.4 complete the proof. \square

The tensor product of two chain complexes (A, d_1) and (B, d_2) , say $A \otimes B$, is formed by taking all products $A_i \otimes B_j$ and letting $(A \otimes B)_k = \bigoplus_{i+j=k} A_i \otimes B_j$. The differential maps are defined as $\partial(a \otimes b) = d_1 a \otimes b + (-1)^i a \otimes d_2 b$, when $a \in A_i$. Then we have $\partial^2 = 0$ and ∂ induces a natural map $\partial : H(A) \otimes H(B) \rightarrow H(A \otimes B)$ such that $\partial(a \otimes b) = a \otimes b$. If $a = d_1 c$ is a boundary and b is a cycle, then $a \otimes b = \partial(c \otimes b)$ is again a boundary which shows that

∂ is well-defined. Let $R = K[x_1, \dots, x_n]$ and $a = (a_1, \dots, a_n)$ be a vector, where all a_ℓ 's are positive integers. Then the a -grading is the graded structure induced by a on R which considers a_ℓ as the degree of x_ℓ for all ℓ . The a -degree of the monomial $m = x_1^{d_1} \cdots x_n^{d_n}$ is $a_1 d_1 + \cdots + a_n d_n$, and the a -degree of a polynomial $f = \sum_{i=1}^r \lambda_i m_i$ denoted by $a(f)$, is the largest a -degree of a monomial in f . Then $\text{in}_a(f) := \lambda_{j_1} m_{j_1} + \cdots + \lambda_{j_k} m_{j_k}$, where $a(f) = a(m_{j_1}) = \cdots = a(m_{j_k})$, and $a(f) > a(m_i)$ for $m_i \neq m_{j_\ell}$.

Lemma 4.6. *Let $I = I_1 + I_2 + \cdots + I_r$ for monomial ideals $I_\ell \subset k[X_\ell]$, where $X_\ell \subset \{x_1, \dots, x_n\}$ and $X_\ell \cap X_{\ell'} = \emptyset$ for all $\ell < \ell'$. Assume that \mathcal{F}_i is the minimal free resolution of R/I_i for all i . Then the minimal free resolution of R/I is obtained by $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_r$.*

Proof. The proof is by induction on r . Assume that $r > 1$. Since differential maps of the tensor complex are defined in terms of differential maps of \mathcal{F}_ℓ 's, the minimality of the tensor complex follows by the minimality of the resolutions of all components. On the other hand, these ideals live in rings with disjoint variables which implies that $\text{Tor}_i(R/(I_1 + \cdots + I_{r-1}), R/I_r) = 0$ for $i > 0$, and so the constructed complex is indeed a minimal free resolution for R/I . \square

Theorem 4.7. *Let $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$ be the clique decomposition of Δ . Assume that \mathcal{F}_i is the minimal free resolution of R/J_{Δ_i} given by the Eagon-Northcott complex for each i . Then the minimal free resolution of R/J_{Δ} is obtained by $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_r$.*

Proof. Assume that $n_\ell = |V(\Delta_\ell)|$ and $t_\ell = \dim(\Delta_\ell)$ for each ℓ . First note that in the case of $n_\ell \leq t_\ell + 1$, the ideal J_{Δ_ℓ} can be identified by the determinantal facet ideal of the $(n_\ell - 1)$ -skeleton of the simplex on $t_\ell + 1$ vertices. Therefore we can get a minimal free resolution of $J_{\Delta'} = J_\Delta$ by Eagon-Northcott complex. So without loss of generality we may assume that $n_\ell \geq t_\ell + 1$ for all ℓ . The proof is by induction on r . Let $r > 1$, $I = J_{\Delta_1} + \cdots + J_{\Delta_{r-1}}$ and $K = J_{\Delta_r}$. By induction hypothesis assume that $F = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \otimes \mathcal{F}_{r-1}$ is a minimal free resolution of R/I , and G a minimal free resolution of K . We consider a vector $a = (a_{11}, a_{12}, \dots, a_{mn})$ in N^{mn} with $a_{11} > a_{12} > \cdots > a_{mn}$. Then we have

$$\text{in}_a([c_1 \cdots c_t | i_1 \cdots i_t]) = \text{in}_<([c_1 \cdots c_t | i_1 \cdots i_t]), \quad \text{in}_a(I) = \text{in}_<(I) \text{ and } \text{in}_a(K) = \text{in}_<(K).$$

Assume that $b = (1, 1, \dots, 1) \in N^{mn}$. Then ideals I and K are both b -homogenous. For all $u \in \text{in}_{<}(I)$ and $v \in \text{in}_{<}(K)$, we have $\text{supp}(u) \cap \text{supp}(v) = \emptyset$ and so the generators of the ideals $\text{in}_{<}(I)$ and $\text{in}_{<}(K)$ are relatively prime which implies $\text{Tor}_i(R/\text{in}_{<}(I), R/\text{in}_{<}(K)) = 0$ for $i > 0$. Now applying [3, Proposition 3.3] we conclude that $\text{Tor}_i(R/I, R/K) = 0$ for $i > 0$ which implies $F \otimes G$ is a resolution for R/J_{Δ} . Since differential maps are defined in terms of differential maps in F and G , we have $\partial_i(F \otimes G)_i \subset \mathfrak{m}(F \otimes G)_{i-1}$ which is equivalent to the minimality of the resolution. \square

Example 4.8. The ideal $I = ([234], [134], [124], [123], [456], [567])$ is the sum of three maximal determinantal ideals. The resolution of each component is given by the Eagon-Northcott complex as follows:

$$\begin{array}{c}
A = \begin{pmatrix} z_1 & -y_1 & x_1 \\ -z_2 & y_2 & -x_2 \\ z_3 & -y_3 & x_3 \\ -z_4 & y_4 & -x_4 \end{pmatrix} \\
0 \longrightarrow R(-4)^3 \longrightarrow R(-3)^4 \xrightarrow{([234] \ [134] \ [124] \ [123])} R \\
 0 \longrightarrow R(-3) \xrightarrow{([456])} R \\
 0 \longrightarrow R(-3) \xrightarrow{([567])} R
\end{array}$$

Then the tensor complex of these resolutions is:

$$0 \longrightarrow R(-10)^3 \xrightarrow{d_3} R(-7)^6 \oplus R(-9)^4 \xrightarrow{d_2} R(-4)^3 \oplus R(-6)^9 \xrightarrow{d_1} R(-3)^6 \xrightarrow{([567] \ [456] \ \cdots \ [123])} R$$

For example, basis elements of the last module $R(-10)^3$ in the resolution are:

$$[1123|1234], [1223|1234], [1233|1234],$$

where by $[ijkl|1234]$ we mean the determinant of the submatrix with row indices i, j, k, ℓ , (not necessarily distinct) and column indices $1, 2, 3, 4$. The differential map acts on the basis element $[1233|1234] \otimes [456] \otimes [567]$ as:

$$\begin{aligned} d_3([1233|1234] \otimes [456] \otimes [567]) &= \partial_{1,1}([1233|1234]) + (-1)^2 \partial_{2,0}([456]) + (-1)^3 \partial_{3,0}([567]) \\ &= (z_1[234] - z_2[134] + z_3[124] - z_4[123]) \otimes [456] \otimes [567] \\ &\quad + [456]([1233|1234] \otimes 1 \otimes [567]) \\ &\quad - [567]([1233|1234] \otimes [456] \otimes 1), \end{aligned}$$

where $\partial_{i,j}$ is the j^{th} differential map in the resolution of the i^{th} ideal.

Lemma 4.6 and Theorem 4.7 together with Lemma 4.5 imply that

Corollary 4.9. *For a closed simplicial complex Δ we have*

$$\beta_{i,j}(R/\text{in}_<(J_\Delta)) = \beta_{i,j}(R/J_\Delta) \quad \text{for all } i, j.$$

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